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Improved Bound Estimates for the Solution of the Discrete Lyapunov Equation

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Abstract

In this research, the work to discrete bounds is extended and improved without the assumption that the system is asymptotically stable. Because attaining the solution itself causes a very massive computational load in case that the dimension of the system matrices is raised, using reasonable bound estimates is fair enough to apply. Thus bounds for the trace and largest eigenvalues are presented and special attention is located on the upper bound estimates for the trace because of their usefulness in robust stability and performance investigation. For the discrete Lyapunov equation, we can achieve some improved upper and lower bound estimates of the solution using the majorization inequality. And the previous research results are improved and generalized with these bounds.

요 약

이 연구에서는 이산 경계치들에 대한 작업이 시스템이 점근적으로 안정하다는 가정없이 확장되고 개선되었다. 솔루션 자체를 얻는 과정이 시스템 행렬들의 차수를 증가시킴에 따라 매우 큰 계산 부담을 발생시키기 때문에 적절한 경계 추정치들을 이용하는 것은 충분히 적용가능하다. 그러므로 트레이스와 최대 고유값에 대한 경계치들이 제시되었고 강인 안정성과 성능 해석에서의 중요성 때문에 특별한 주의가 트레이스의 상한 경계치에 주어졌다. 이산시간 리아푸노프 방정식에 대해 주요화 부등식을 이용하여 솔루션의 상한과 하한 경계 추정치들을 얻을 수 있다. 그리고 이전의 연구결과들을 이 경계치들로 일반화하고 개선하였다.

Keywords
bound estimates, discrete Lyapunov equation, similarity transformation, majorization inequality

Ⅰ. Introduction

The continuous and discrete Lyapunov equations have been generally employed in diverse fields of engineering and control theory. In many applications of signal processing and control system analysis, it is necessary to estimate bounds of the solutions for these equations [1]. As one of the research work for control system analysis, Y.-M. Kim and W.-B. Baek extended the research to adaptive sliding mode control for quadrotor with feedback linearization method [2]. Including this study, numerous jobs have been devoted to estimate the extent or size of the solutions for these equation during the past three decades [2]-[5]. However, the literature for this topic has had some deficiency. For the discrete Lyapunov equation, most of the upper bounds and lower estimates of the solution are established on the assumption of $\lambda_1(AA^T)<1$ unfortunately [4]-[18]. By utilizing the similarity transformation [3], firstly provided some

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discrete upper bound estimates not having the presumption that the system is asymptotically stable, $i.e.$, $\lambda_1(AA^T) < 1$. Lee *et al.* investigated some bounds for unified and continuous Lyapunov matrix equation [6][7]. Hyun et al. performed the bound estimates for stochastic unified system [8]. In this paper, by adopting the virtue of the similarity transformation of [3], new and fine bound estimates of the solution are developed without the assumption of $\lambda_1(AA^T) < 1$ for the discrete Lyapunov equation. These upper bounds are tighter than the results of [3]. Furthermore, we also obtain some new lower bounds. In addition, we give some numerical examples to demonstrate the merit of the proposed results.

II. Main Results

2.1 Preliminary Results

In the next, $R^{n\times n}$ is represented for the set of $n\times n$ real matrices. For $A\in R^{n\times n}$, let $tr(A)$ denote the trace of A. Suppose $A\in R^{n\times n}$ be an random symmetric matrix, afterward we presume that the eigenvalues of A are organized so that $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$.

Lemma 1[19, p7-12]: If the components of x and y are nonnegative integers, then the following conditions are equivalent:

(i) x can be derived from y by a finite number of transfers.

(ii) the sum of the k largest components of x is less than or equal to the sum of the k largest components of $y, k=1,2, \ldots, n$, with equality when $k=n$.

(iii)

$$\sum_{\pi} \alpha_{\pi(1)}^{x_1}\alpha_{\pi(2)}^{x_2} \cdots \alpha_{\pi(n)}^{x_n} \le \sum_{\pi} \alpha_{\pi(1)}^{y_1}\alpha_{\pi(2)}^{y_2} \cdots \alpha_{\pi(n)}^{y_n} \tag{1}$$

whenever each $\alpha_i > 0$. Here Σ_π denotes summation over all permutations.

For $x, y\in R^n$,

$$x < y \text{ if } \sum_{i=1}^{k} x[i] \le \sum_{i=1}^{k} y[i], \ k=1,2, \ldots, n \tag{2}$$

when $x<y$, x is said to be majorized by y. Then x is called weakly majorized by y and x is signaled by $x <_w y$. This notation and terminology was introduced by Hardy, Littlewood, and Polya [20]. The next lemmas are employed to demonstrate the principal results.

Lemma 2[21, p49]: If $A, B\in R^{n\times n}$ are symmetric matrices and $1 \le i_1 < \cdots < i_k \le n$, then for $k=1$, $2, \cdots, n$,

$$\sum_{t=1}^{k} \lambda_{i_t}(A+B) \le \sum_{t=1}^{k} \lambda_{i_t}(A) + \sum_{t=1}^{k} \lambda_{i_t}(B) \tag{3}$$

Especially, we have

$$\sum_{i=1}^{k} \lambda_i(A+B) \le \sum_{i=1}^{k} [\lambda_i(A) + \lambda_i(B)] \tag{4}$$

with equality when $k=n$.

Lemma 3[21, p49]: If $A, B\in R^{n\times n}$ are symmetric matrices and $1 \le i_1 < \cdots < i_k \le n$, then for $k=1$, $2, \cdots, n$,

$$\sum_{t=1}^{k} \lambda_{i_t}(A+B) \le \sum_{t=1}^{k} \lambda_{i_t}(A) + \sum_{t=1}^{k} \lambda_{n-t+1}(B) \tag{5}$$

Especially, we have

$$\sum_{i=1}^{k} \lambda_{n-i+1}(A+B) \le \sum_{i=1}^{k} [\lambda_{n-i+1}(A) + \lambda_{n-i+1}(B)] \tag{6}$$

with equality when $k=n$.

Lemma 4[21, p48]: If $A, B \in R^{n \times n}$ are symmetric positive semidefinite matrices and $1 \le i_1 < \cdots < i_k \le n$, then for $k = 1, 2, \cdots, n$,

$$\sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_{n-t+1}(B) \le \sum_{t=1}^{k} \lambda_{i_t}(AB) \le \sum_{t=1}^{k} \lambda_{i_t}(A)\lambda_t(B) \tag{7}$$

Especially, we have

$$\sum_{i=1}^{k} \lambda_{n-i+1}(A)\lambda_{n-i+1}(B) \le \sum_{i=1}^{k} \lambda_{n-i+1}(AB) \le \sum_{i=1}^{k} \lambda_{n-i+1}(A)\lambda_i(B) \tag{8}$$

and

$$\sum_{i=1}^{k} \lambda_i(A)\lambda_{n-t+1}(B) \le \sum_{i=1}^{k} \lambda_i(AB) \le \sum_{i=1}^{k} \lambda_i(A)\lambda_i(B) \tag{9}$$

with equality when $k = n$.

Lemma 5[19, p160, B.7 eq(9)]: If $x_1 \ge \cdots \ge x_n$, $y_1 \ge \cdots \ge y_n$ and $x <_w y$, then for any real array $u_1 \ge \cdots \ge u_n \ge 0$,

$$\sum_{i=1}^{k} x_i u_i \le \sum_{i=1}^{k} y_i u_i, \quad k = 1, 2, \cdots, n \tag{10}$$

Lemma 6[12]: Let $A \in R^{n \times n}$, and $A = U^T \Lambda U$ where U is orthogonal and Λ is diagonal with $0 \le \lambda_i(\Lambda) < 1$. Then

$$(I - A)^{-1} = I + A + A^2 + \cdots \tag{11}$$

2.2 The upper bound estimates of the solution for Lyapunov equation

Consider the discrete Lyapunov equation

$$P - A^T P A - Q = 0 \tag{12}$$

For making up for the example that $\lambda_1(AA^T)$ is not within the circle of radius 1, we should establish the next alternation. Applying the similarity conversion [6 eq.(14)]-[21 p56], we set

$$\widetilde{P} = U^T P U, \ \widetilde{Q} = U^T Q U, \ \widetilde{A} = U^{-1} A U \tag{13}$$

Then the modified Lyapunov equation is obtained

$$\widetilde{P} - \widetilde{A}^T \widetilde{P} \widetilde{A} - \widetilde{Q} = 0 \tag{14}$$

where $\lambda_1(\widetilde{A}\widetilde{A}^T) < 1$.

Then the previous works for bound estimates is valid. Using equation (13) and the above lemmas, the next theorems can be attained.

Theorem 1: If the discrete Lyapunov equation (12) is applied and $\lambda_1(\widetilde{A}\widetilde{A}^T) < 1$, then we attain the following inequality as

$$\sum_{i=1}^{k} \lambda_i(P) \le \sum_{i=1}^{k} \frac{\lambda_i(E^{-1}Q)\lambda_i(E)}{\lambda_{n-i+1}(I - \widetilde{A}\widetilde{A}^T)} \tag{15}$$

where $E = U^{-T}U^{-1}$ and U is the transformation matrix in equation (13).

<Proof> From Komaroff's work [12], the solution to equation (14), using integer l, is

$$\widetilde{P} = \sum_{l=0}^{\infty} (\widetilde{A}^T)^l \widetilde{Q} \widetilde{A}^l = \widetilde{Q} + \widetilde{A}^T \widetilde{Q} \widetilde{A} + (\widetilde{A}^T)^2 \widetilde{Q} \widetilde{A}^2 + \cdots \tag{16}$$

For notational convenience, set $\widetilde{B} = \widetilde{A}\widetilde{A}^T$, then

$$\lambda_i\left((\widetilde{A}^T)^l \widetilde{Q} \widetilde{A}^l\right) = \lambda_i(\widetilde{Q}\widetilde{B}^l) \tag{17}$$

Applying Lemma 2 to equation (16), in view of equation (17) and Lemma 4, we have

$$\sum_{i=1}^{k}\lambda_i(\widetilde{P})\leq \sum_{i=1}^{k}\left[\lambda_i(\widetilde{Q})+\lambda_i(\widetilde{Q}\widetilde{B}^2)+\cdots\right] \\ \leq \sum_{i=1}^{k}\lambda_i(\widetilde{Q})\left[1+\lambda_i(\widetilde{B})+\lambda_i(\widetilde{B}^2)+\cdots\right] \tag{18}$$

Considering $\lambda_i(\widetilde{B})<1$, Lemma 6 can be employed in equation (18) to obtain

$$\sum_{i=1}^{k}\lambda_i(\widetilde{P})\leq \sum_{i=1}^{k}\lambda_i(\widetilde{Q})\left[1-\lambda_i(\widetilde{A}\widetilde{A}^T)\right]^{-1} \tag{19}$$

From $\widetilde{P}= U^TPU$ and $\widetilde{Q}= U^TQU$, it is easy to know that

$$\lambda_i(\widetilde{P})=\lambda_i(U^TPU)=\lambda_i(UU^TP)=\lambda_i(E^{-1}P) \tag{20}$$

and

$$\lambda_i(\widetilde{Q})=\lambda_i(U^TQU)=\lambda_i(UU^TQ) \\ =\lambda_i(E^{-1}Q) \tag{21}$$

Then equation (19) can be rewritten as follows:

$$\sum_{i=1}^{k}\lambda_i(E^{-1}P)\leq \sum_{i=1}^{k}\lambda_i(E^{-1}Q)\left[1-\lambda_i(\widetilde{A}\widetilde{A}^T)\right]^{-1} \tag{22}$$

From

$$\left[1-\lambda_1(\widetilde{A}\widetilde{A}^T)\right]^{-1}\geq \left[1-\lambda_2(\widetilde{A}\widetilde{A}^T)\right]^{-1}\geq\cdots \\ \geq \left[1-\lambda_n(\widetilde{A}\widetilde{A}^T)\right]^{-1} \tag{23}$$

and

$$\lambda_1(E^{-1}Q)\geq\lambda_2(E^{-1}Q)\geq\cdots\geq\lambda_n(E^{-1}Q) \tag{24}$$

we can obviously recognize that

$$\lambda_1(E^{-1}Q)\left[1-\lambda_1(\widetilde{A}\widetilde{A}^T)\right]^{-1}\geq \\ \lambda_2(E^{-1}Q)\left[1-\lambda_2(\widetilde{A}\widetilde{A}^T)\right]^{-1}\geq\cdots\geq \\ \lambda_n(E^{-1}Q)\left[1-\lambda_n(\widetilde{A}\widetilde{A}^T)\right]^{-1} \tag{25}$$

For notational convenience, set $x_i=\lambda_i(E^{-1}P)$ and $y_i=\lambda_i(E^{-1}Q)\left[1-\lambda_i(\widetilde{A}\widetilde{A}^T)\right]^{-1}$. From equation (22) and (25), we know

$$x=(x_1,x_2,\cdots,x_n)<_w y=(y_1,y_2,\cdots,y_n) \tag{26}$$

Since $\lambda_1(E)\geq\lambda_2(E)\geq\cdots\geq\lambda_n(E)\geq 0$, in view of Lemma 5, we have

$$\sum_{i=1}^{k}\lambda_i(E^{-1}P)\lambda_i(E)\leq\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q)\lambda_i(E)}{1-\lambda_i(\widetilde{A}\widetilde{A}^T)} \tag{27}$$

Applying Lemma 4 to equation (27), we have

$$\sum_{i=1}^{k}\lambda_i(P)=\sum_{i=1}^{k}\lambda_i(PEE^{-1})=\sum_{i=1}^{k}\lambda_i(E^{-1}PE) \\ \leq\sum_{i=1}^{k}\lambda_i(E^{-1}P)\lambda_i(E)\leq\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q)\lambda_i(E)}{1-\lambda_i(\widetilde{A}\widetilde{A}^T)} \tag{28}$$

Note that $1-\lambda_i(\widetilde{A}\widetilde{A}^T)=\lambda_{n-i+1}(I-\widetilde{A}\widetilde{A}^T)$. This completes the proof.

Corollary 1: For the discrete Lyapunov equation (12), if $\lambda_1(\widetilde{A}\widetilde{A}^T)<1$, then we have

$$\sum_{i=1}^{k}\lambda_i(P)\leq\lambda_1(E)\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q)}{\lambda_{n-i+1}(I-\widetilde{A}\widetilde{A}^T)} \\ when\ k=n, \\ tr(P)\leq\lambda_1(E)\sum_{i=1}^{n}\frac{\lambda_i(E^{-1}Q)}{\lambda_{n-i+1}(I-\widetilde{A}\widetilde{A}^T)} \tag{29}$$

Furthermore, we have

$$tr(P)\leq\frac{\lambda_1(E)}{1-\lambda_1(\widetilde{A}\widetilde{A}^T)}\sum_{i=1}^{n}\lambda_i(E^{-1}Q) \tag{30}$$

<Remark> The bounds of equation (29) and (30) were presented in [3]. Therefore the result of [3] can be generalized by Theorem 1. And Theorem 1 also offers a improved bound estimate of the solution.

Theorem 2: For the discrete Lyapunov equation (11), we have

$$\prod_{i=1}^{k}\lambda_i(P) \le \left[\frac{1}{k}\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q)\lambda_i(E)}{\lambda_{n-i+1}\left(I-\widetilde{A}\widetilde{A}^T\right)}\right]^k \tag{31}$$

<Proof> Using the arithmetic-mean geometric-mean inequality [19, p125-126] on equation (15) leads directly to the above bound. If $\lambda_i(P)>0$ for $i=1,\ ...,n$, then

$$\left(\prod\lambda_i(P)\right)^{1/n} \le \sum\frac{\lambda_i(P)}{n} \tag{32}$$

equation (32) leads equation (15) to equation (31) as follows:

$$\begin{aligned}\prod_{i=1}^{k}\lambda_i(P) &\le \sum_{i=1}^{k}\lambda_i(P)\\ &\le \left[\frac{1}{k}\sum_{i=1}^{k}\frac{\lambda_i(E^{-1}Q)\lambda_i(E)}{\lambda_{n-i+1}(I-\widetilde{A}\widetilde{A}^T)}\right]^k\end{aligned} \tag{33}$$

2.3 The lower bound estimates of the solution for Lyapunov equation

In this section, by utilizing the majorization transformation, some lower bound estimates of the solution can be obtained for the discrete Lyapunov equation without the presumption of $\lambda_1(AA^T)<1$.

Theorem 3: Let the positive definite matrix P be the solution to equation (12). If $\lambda_1\left(AA^T\right)<1$, then we have

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_{n-i+1}(P) \ge \frac{\lambda_n(E)}{\lambda_1(I-\widetilde{A}\widetilde{A}^T)}\sum_{i=1}^{k}\lambda_{n-i+1}(E^{-1}Q)\\ &tr(P) \ge \frac{\lambda_n(E)}{\lambda_1(I-\widetilde{A}\widetilde{A}^T)}tr(E^{-1}Q)\end{aligned} \tag{34}$$

<Proof> For $\widetilde{P}=\widetilde{A}^T\widetilde{P}\widetilde{A}+\widetilde{Q}$, we have

$$\sum_{i=1}^{k}\lambda_{n-i+1}(\widetilde{P})=\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{A}^T\widetilde{P}\widetilde{A}+\widetilde{Q}\right) \tag{35}$$

By Lemma 3, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{P}\right)\ge\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{A}^T\widetilde{P}\widetilde{A}\right)\\ &+\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{Q}\right)=\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{A}\widetilde{A}^T\widetilde{P}\right)\\ &+\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{Q}\right)\end{aligned} \tag{36}$$

Applying Lemma 4 to equation (36), we have

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_{n-i+1}(\widetilde{P})\ge\\ &\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{A}\widetilde{A}^T\right)\lambda_{n-i+1}\left(\widetilde{P}\right)+\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{Q}\right)\end{aligned} \tag{37}$$

From equation (37), we know

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_{n-i+1}(\widetilde{P})\ge\\ &\lambda_n\left(\widetilde{A}\widetilde{A}^T\right)\sum_{i=1}^{k}\lambda_{n-i+1}(\widetilde{P})+\sum_{i=1}^{k}\lambda_{n-i+1}(\widetilde{Q})\end{aligned} \tag{38}$$

Then

$$\left[1-\lambda_n\left(\widetilde{A}\widetilde{A}^T\right)\right]\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{P}\right)\ge\sum_{i=1}^{k}\lambda_{n-i+1}\left(\widetilde{Q}\right) \tag{39}$$

Considering $\lambda_n\left(\widetilde{A}\widetilde{A}^T\right)<1$, from equation (20) and (21), equation (39) can be rewritten as follows:

$$\sum_{i=1}^{k}\lambda_{n-i+1}(E^{-1}P) \geq \frac{1}{1-\lambda_n\left(\widetilde{A}\widetilde{A}^T\right)}\sum_{i=1}^{k}\lambda_{n-i+1}(E^{-1}Q) \quad (40)$$

In view of Lemma 4, we have

$$\sum_{i=1}^{k}\lambda_{n-i+1}(E^{-1}P) \leq \sum_{i=1}^{k}\lambda_{n-i+1}(P)\lambda_i(E^{-1}) \leq \lambda_1(E^{-1})\sum_{i=1}^{k}\lambda_{n-i+1}(P) \quad (41)$$

Applying equation (41) to equation (40), we obtain

$$\sum_{i=1}^{k}\lambda_{n-i+1}(P) \geq \frac{\lambda_n(E)}{1-\lambda_n(\widetilde{A}\widetilde{A}^T)}\sum_{i=1}^{k}\lambda_{n-i+1}(E^{-1}Q) \quad (42)$$

when $k=n$, we have

$$tr(P) \geq \frac{\lambda_n(E)}{\lambda_1(I-\widetilde{A}\widetilde{A}^T)}tr(E^{-1}Q) \quad (43)$$

Note that $\lambda_1(E^{-1})=\lambda_n^{-1}(E)$. This completes the proof.

III. Numerical Examples

3.1 Example 1

Consider the discrete Lyapunov equation (12) with system matrix A in [6 eq.(42)]

$$A=\begin{bmatrix}0.7563 & 0.8564\\ 0.3536 & -0.7856\end{bmatrix},\ Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Then the eigenvalues of AA^T are given by

$$\lambda_1(AA^T)=1.5174 \quad \lambda_2(AA^T)=0.5302 \quad (44)$$

In order to overcome the difficulty that $\lambda_1(AA^T)$ is unstable, we introduce the similarity transformation matrix U. For this example, we choose $U=\begin{bmatrix}1.9359 & 0.2045\\ 0.2312 & -1.1352\end{bmatrix}$. Afterward we can have the Jordan-converted matrix and its eigenvalues as

$$\widetilde{A}\widetilde{A}^T=\begin{bmatrix}0.8915 & 0.0654\\ 0.0654 & 0.9072\end{bmatrix} \quad \lambda_1\left(\widetilde{A}\widetilde{A}^T\right)=0.9653,\ \lambda_2\left(\widetilde{A}\widetilde{A}^T\right)=0.8335 \quad (45)$$

Now, we eliminate the assumption of $\lambda_1\left(\widetilde{A}\widetilde{A}^T\right)<1$. Using Theorem 1, we have

$$\lambda_1(P) \leq 82.9450,\ tr(P) \leq 85.0316 \quad (46)$$

Using Theorem 2, we obtain the following determinant bound

$$|P| \leq 84.0335 \quad (47)$$

From Theorem 3, we have

$$\lambda_n(P) \geq 2.0867,\ tr(P) \geq 8.0919 \quad (48)$$

Using Theorem 1 of [3], we have

$$tr(P) \leq 111.7665 \quad (49)$$

Using Theorem 2 of [3], we have

$$tr(P) \leq 88.9502 \quad (50)$$

By comparison, we know equation (46) is better than equation (49) and (50).

3.2 Example 2

Consider the discrete Lyapunov equation (12) with

$$A=\begin{bmatrix}0.2539 & 0.5810\\ 0 & 0.8958\end{bmatrix},\ Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Then the eigenvalues of AA^T are given by

$$\lambda_1(AA^T) = 1.1599 \qquad (51)$$
$$\lambda_2(AA^T) = 0.0446$$

In order to overcome the difficulty that $\lambda_1(AA^T)$ is unstable, we introduce the similarity transformation matrix U. For this example, we choose $U = \begin{bmatrix} 0.9232 & 0.0787 \\ 0.2141 & -0.2140 \end{bmatrix}$. Afterward we can have the Jordan-converted matrix and its eigenvalues as

$$\tilde{A}\tilde{A}^T = \begin{bmatrix} 0.2141 & -0.3262 \\ -0.3262 & 0.7387 \end{bmatrix} \qquad (52)$$
$$\lambda_1(\tilde{A}\tilde{A}^T) = 0.8950,\ \lambda_2(\tilde{A}\tilde{A}^T) = 0.0578$$

Now, we eliminate the assumption of $\lambda_1(\tilde{A}\tilde{A}^T) < 1$. Using Theorem 1, we have

$$\lambda_1(P) \le 167.3835,\ tr(P) \le 167.4438 \qquad (53)$$

By Theorem 2, we obtain the following determinant bound

$$|P| \le 167.3844 \qquad (54)$$

From Theorem 3, we have

$$\lambda_n(P) \ge 0.0604,\ tr(P) \ge 1.1217 \qquad (55)$$

Using Theorem 1 of [3], we have

$$tr(P) \le 176.9051 \qquad (56)$$

Using Theorem 2 of [3], we obtain

$$tr(P) \le 168.4448 \qquad (57)$$

By comparison, we know equation (53) is better than equation (56) and (57).

Ⅳ. Conclusions

One of essential problem for system stability analysis is the evaluation of the solution for the Lyapunov equation. Inspired by this issue, in the coverage of this paper, investigation of stability analysis for the solution of discrete Lyapunov equation is performed. By using similarity transformation of [3], we have proposed some new bounds which are extended with removal of the assumption of $\lambda_1(AA^T) < 1$. Compared with the results of [3], the obtained upper bounds are tighter. The results of [3] shows that $tr(P) \le 111.7665$, $tr(P) \le 88.9502$. But, the results of this study shows that $tr(P) \le 85.0316$. Then, we have the better results for upper bounds of trace of the solution of Lyapunove equation. Moreover, we give fine lower bounds. The work results illustrates that $\lambda_n(P) \ge 2.0867$, $tr(P) \ge 8.0919$. These lower bounds are new.

For verifying these outcomes, we demonstrate that mathematical evidence for better bound estimates of solution for the discrete Lyapunov equation from the numerical examples. Besides, performing the research for robust stability with some existing works would be valuable as future study. D. G. Lee[22][23] developed the works for robust stability of state feedback control with linear perturbation. As another work in stability analysis, the robust controller design for computer-controlled unified system is suggested by D. Lee and W. Lee [24]. Thus, the future study would be extended to the works for that of output feedback control for discrete-time and unified system. Another possible future research topic can be started from the research results of D.-G. Lee and I. Hyun [25]. They investigated the stochastic optimal controller design of decentralized singularly perturbed unified system. We consider this work would be extended to the study topic of feedback controller design with the case of linear perturbation.

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